Final- Complex Analysis (2022-23)
Time: 3 hours.
Attempt all questions, giving proper explanations.

You may quote any result proved in class without proof (unless, of course, you are asked to prove the result).

1. Suppose $f = u + iv$ is holomorphic in a region Ω.
   (a) Show that ∇u and ∇v are orthogonal. **[2 marks]**
   (b) Suppose also that $\bar{f}$ is holomorphic in Ω. Show that f is a constant. **[2 marks]**

2. Suppose the continuous function $f(e^{i\theta})$ on the unit circle satisfies $|f(e^{i\theta})| \leq M$ for all $\theta \in [0, 2\pi)$, and $|\int_{|z|=1} f(z)dz| = 2\pi M$. Show that $f(z) = c\bar{z}$ for some constant c with modulus $|c| = M$. **[6 marks]**

3. Let $a > 0$. Compute
$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+1)(x^2-2x+2)}\, dx. \qquad \textbf{[8 marks]}$$
[Hint: Consider an upper half disk]

4. Let
$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(w) = i\frac{1-w}{1+w}.$$
Show that $F : \mathbb{H} \to \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \to \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\mathbb{D}$ is the open unit disc. **[6 marks]**

5. Let $h : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies
$$|h(z)| \leq 1 \quad \text{and} \quad h(i) = 0.$$
Prove that
$$|h(z)| \leq \left|\frac{z-i}{z+i}\right| \quad \text{for all } z \in \mathbb{H}. \qquad \textbf{[6 marks]}$$

6. Given two points $z_1, z_2 \in \mathbb{D}$, show that there exists an automorphism ϕ such that $\phi(z_1) = 0$ and $\phi(z_2) = s$ for some real number $s \in [0, 1)$. **[6 marks]**

7. Let $f_k : \mathbb{D} \to \mathbb{C}$ be a sequence of functions such that $|f_k(z)| \leq 1$ for all $z \in \mathbb{D}$ and all k. Let n be a positive integer. Find a constant $C_n > 0$ such that
$$\left|f_k^{(n)}(z) - f_k^{(n)}(w)\right| \leq C_n|z-w| \quad \text{for all } k \text{ and all } z, w \in \mathbb{B}\left(0, \frac{1}{2}\right),$$
where $\mathbb{B}\left(0, \frac{1}{2}\right)$ is the open disc of radius $\frac{1}{2}$ around 0. **[6 marks]**

8. Let f be an entire function and let $n(r)$ denote the number of zeroes of f inside the disc $B(0, r)$ of radius r centered around 0. Suppose there exist constants $0 < c < C < \infty$ and $\rho > 0$, $r_0 > 0$ such that
$$cr^\rho \leq n(r) \leq Cr^\rho \tag{0.1}$$
for all $r > r_0$. Let $z_1, z_2, \cdots$ denote the *non-zero* zeroes of f.
   (a) Show for $s > \rho$ **[3 marks]**
$$\sum_k \frac{1}{|z_k|^s} < \infty. \tag{0.2}$$
   (b) Show for $s < \rho$ **[3 marks]**
$$\sum_k \frac{1}{|z_k|^s} = \infty. \tag{0.3}$$
   (c) For the function $f(z) = \sin(\pi z)$ verify that (0.1) holds for some $\rho > 0$, and verify (0.2) and (0.3) . **[2 marks]**